ART. XVII. — *On "Confocal Quadrics of Moments of Inertia" pertaining to all Planes in Space, and Loci and Envelopes of Straight Lines whose "Moments of Inertia" are Constant.*

By MARTIN GARDINER, C.E.

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ABSTRACT.

The author commences by solving the following problem, by the Cartesian co-ordinate method :—

Problem.—Given any number of points $P_1, P_2, P_3, \ldots$ in space, and corresponding numbers $a_1, a_2, a_3, \ldots$, known in signs and magnitudes as respective multipliers; to find the Envelope of a plane $L\,L\,L$, such that, in every position it can assume, we shall have

$$a_1 \cdot p_1^2 + a_2 \cdot p_2^2 + a_3 \cdot p_3^2 + \ldots = S,$$

in which $p_1^2, p_2^2, p_3^2, \ldots$, represent the squares of the pedals from the points $P_1, P_2, P_3, \ldots$, to the plane $L\,L\,L$, and S a constant entity known in sign and magnitude.

He finds the equation of the envelope of the plane $L\,L\,L$ to be that of a Quadric whose centre is coincident with the mean-centre of the given points for the multipliers $a_1, a_2, a_3, \ldots$ And from the *form* of the equation arrived at (which is given abridged and expanded), he infers that for all possible values of the entity S, the corresponding Quadrics are Confocal Quadrics.

He then shows by a purely geometrical method (independent of co-ordinates) that for any constant value of S, the

envelope of the plane $L\,L\,L$ is a Quadric whose centre is coincident with the mean centre of the points P_1, P_2, P_3, . . . , and their respective multipliers a_1, a_2, a_3, ... And he shows that the quadrics corresponding to all possible values of the entity S, are Confocal Quadrics.

In order to amplify his Geometrical Method, he proceeds to give a full and complete solution to the particular cases in which the given points P_1, P_2, P_3, . . . , are all in one straight line. And he shows that it depends on the state of the data, as to whether the Confocal Quadrics be Ellipsoids; Hyperboloids of One Sheet; Hyperboloids of Two Sheets; Spheres; or Paraboloids.

He then directs attention to the Physical Aspect of the problem, which he enunciates as follows:—

Problem.—Given any masses M_1, M_2, M_3, . . . , in space, and corresponding units a_1, a_2, a_3, . . . , known in signs as their respective multipliers; to find the Envelope of a plane $L\,L\,L$, such that in every position it can assume, we shall have the sum of the Moments of Inertia of the masses represented by

$$a_1 \,.\, \lessgtr m_1\,(P_1 L)^2 \;+\; a_2 \,.\, \lessgtr m_2\,(P_2 L)^2 \;+\; a_3 \,.\, \lessgtr m_3 \,.\, (P_3 L)^2 \;+\; . \;.\; . \;=\; a \text{ constant } S,$$

in which m_1, m_2, m_3, . . . represent molecules of the masses M_1, M_2, M_3, . . . , at any points P_1, P_2, P_3, . . . in those masses, and in which $P_1 L$, $P_2 L$, $P_3 L$, . . . represent the pedals from the points P_1, P_2, P_3, . . . , to the plane $L\,L\,L$.

In elucidation of this aspect of the problem, he reconsiders the particular cases, in which he now replaces the given points or molecules at P_1, P_2, P_3, . . . all in one line, by Spheres whose centres are all in one straight line. He shows that the results arrived at previously, apply when masses replace mere molecules; and that, according to analogous states of the data, the Confocal Quadrics will be Ellipsoids, Hyperboloids, Spheres, or Paraboloids.

He establishes the limiting values for the constant S, and exposes the limiting forms of the Quadrics in minute and full detail. And he corroborates a remarkable theorem of

Duhamel's, as to the existence of two points, *for each of which* Poinsot's "Ellipsoid of stress" is a Sphere. He shows, moreover, these two points to belong to a "Focal Conic" of the family of Confocal Quadrics.

In the case in which the bodies are Spheres *situated in any manner in space,* he gives a simple and effective method of finding the three principal axes of inertia.

He then records the following eight Theorems, as results of his investigations:—

Theorem 1.

Given any masses M_1, M_2, M_3, . . . in space, and corresponding numbers a_1, a_2, a_3, . . . of known signs as multipliers. If a plane $L\ L\ L$ (otherwise unrestricted) be such that in every position it can assume, the sum of the moments of inertia of the entities $a_1 . M_1$, $a_2 . M_2$, $a_3 . M_3$, . . . , with respect to it, be of any constant magnitude S, then will the envelope of the plane be a determinable quadric Q, whose centre is coincident with the mean centre of the entities. And the whole system of quadrics Q_1, Q_2, Q_3, ... corresponding to all values S_1, S_2, S_3, ... , of S, will be concentric, coaxial, and confocal quadrics. And in all cases in which the multipliers a_1, a_2, . . . are all positive, the quadrics will be Ellipsoids and Hyperboloids of One Sheet.

Theorem 2.

Given any masses M_1, M_2, M_3, . . . in space, and corresponding numbers a_1, a_2, a_3, . . . of known signs, as multipliers. The *envelope* of all planes $L\ L\ L$ passing through any given point V in space, and such that the sum of the moments of inertia of the entities $a_1 . M_1$, $a_2 . M_2$, $a_3 . M_3$, ... , with respect to them severally, is of any constant magnitude S, will be a determinable quadric cone C, which envelopes a determinable quadric Q whose centre is coincident with the mean centre O of the entities. And the whole family of such cones C_1, C_2, C_3, . . . , corresponding to all values S_1, S_2, S_3, . . . , of S, will be coaxial and confocal cones enveloping coaxial and confocal quadrics, whose common centre is the mean centre O of the entities

$a_1. M_1$, $a_2. M_2$, ... And if the point V be at infinity, and given in direction by means of a vector $O\ R$ passing through the mean centre O; then, corresponding to various values of S, the envelopes of $L\ L\ L$ consist of a system of confocal cylinders enveloping the quadrics, and having as common principal axis the directing vector $O\ R$.

Now M_1, M_2, M_3, . . . being masses, and a_1, a_2, a_3, numbers known in signs: we know that if a plane $L\ L\ L$ be such that the sum of the moments of inertia of the entities $a_1. M_1$, $a_2. M_2$, $a_3. M_3$, . . . , with respect to it is of a constant magnitude S, then will the envelope of the plane be a determinable quadric Q. But the line of intersection $l\ l$ of any two mutually orthogonal planes, both tangent to the quadric Q, is obviously such that the sum of the moments of inertia of the entities with respect to it is represented by $2.s$.

We can easily form the equations of tangent planes to the quadric Q, and express their mutual orthogonism; but we need not try to evolve an equation of *a surface* which could be the envelope of all the lines $l\ l$ of intersection of the pairs of mutually orthogonal tangent planes to Q. This is obvious:—for if we suppose p to be any point whatever on any surface, and construct a Poinsot Ellipsoid having such point as centre, we perceive that the lines $l\ l$ through the point form a cone, and cannot generally *all* be tangents at one point to any other surface. However, we proceed to find the Loci and Envelopes of lines $l_1\ l_1$ which fulfil the conditions as to equality of moments of inertia, and respecting which other conditions are imposed.

1°.—With respect to all the lines $l_1\ l_1$ which are parallel to any fixed straight line $R\ R$ passing through the mean centre O, which is also the centre of the quadric Q_1.

If through O we draw a plane normal to the line $R\ R$, and that we put $c_1\ c_1\ c_1$ to represent the conic which constitutes its trace on the quadric Q_1: then, from a well-known theorem, we perceive that the pairs of mutually orthogonal tangent planes whose points of contact lie in the conic $c_1\ c_1\ c_1$, give us all the lines $l_1\ l_1$ parallel to the fixed line $R\ R$, and that they constitute a Right Circular Cylinder having $R\ R$ as central axis.

2°.—With respect to all the lines $l_1 l_1$ situated in tangent planes to the quadric Q_1.

We may first observe that if $P_1 P_1 P_1$ be any fixed plane tangent to the quadric Q_1, and that we project the quadric itself orthogonally by means of other tangent planes upon $P_1 P_1 P_1$, then will the projection be a conic $c_1 c_1 c_1$ situated in the plane $P_1 P_1 P_1$, which is obviously the envelope of all the lines $l_1 l_1$ in the plane.

3°.—With respect to all the lines $l_1 l_1$ situated in any plane $B B B$ whatever.

We first proceed and find the sum s_q of the moments of inertia of the entities $a_1 . M_1$, $a_2 . M_2$, $a_3 . M_3$, . . . , with respect to the plane $B B B$. We then find the quadric Q_o such that the sum of the moments of inertia of the entities with respect to any of its tangent planes is $= 2.s_1, - s_q$. Then, obviously, the orthogonal projection of the quadric Q_o so found (by means of tangent planes to it) upon the plane $B B B$ will be a conic, which is the envelope of the lines $l_1 l_1$ situated in the plane.

The following is an obvious deduction :—

Theorem 3.

Given any masses M_1, M_2, M_3, . . . in space, and corresponding numbers a_1, a_2, a_3, . . . of known signs as multipliers; and given also the system of confocal quadrics Q_1, Q_2, Q_3, . . . , such that the sum of the moments of inertia of the entities $a_1 . M_1$, $a_2 . M_2$, $a_3 . M_3$, . . . , with respect to tangent planes to the quadrics are equals respectively to s_1, s_2, s_3, . . . ; then the orthogonal projections of the quadrics on any given plane $B B B$ in space, constitute a family of confocal conics, which are the respective envelopes of straight lines $l_1 l_1$, $l_2 l_2$, $l_3 l_3$, ... , situated in the plane, such that the sum of the moments of inertia of the entities $a_1 . M_1$, $a_2 . M_2$, $a_3 . M_3$, . . . , with respect to them, are determinable constants. And if the plane $B_1 B_1 B$ be parallel to either one of the two systems of parallel circular sections of the confocal quadrics, then will the projections of the quadrics on the plane be a system of concentric circles.

NOTE.—The differences of the moments of inertia with respect to the lines $l_1 l_1$, $l_2 l_2$, $l_3 l_3$, . . . , (tangents to the respective conics) on the plane $B B B$ are obviously equals to the differences of the moments of inertia with respect to tangent planes to the quadrics Q_1, Q_2, Q_3, . . .

If we draw planes $P_1 P_1 P_1$, $P_2 P_2 P_2$, . . . , through any diameter $D D$ of any one Q of the family of Confocal quadrics. the lines $l\,l$ situated in these planes and such that the sum of the moments of inertia of the entities $a_1 . M_1$, $a_2 . M_2$, $a_3 . M_3$, . . . , with respect to them, severally, is of any constant magnitude $2.s$, have (as already observed) as envelopes, in the planes, determinable conics. And we know that those of the lines $l\,l$ which are parallel to $D D$ form a circular cylinder, having the line $D D$ as axis. But it is easy to perceive that it is only when the axis $D D$ is normal to one of the circular sections of the quadric Q that the conics cut $D D$ in the one and same point, at which the lines $l\,l$ form a tangent plane to all the conics. Hence:—

THEOREM 4.

Given any number of masses M_1, M_2, M_3, . . . , in space, and corresponding numbers a_1, a_2, a_3, . . . , of known signs as multipliers; if a straight line $l\,l$ move in space so as to be always in contact with the line $D D$ of a diameter of any quadric Q (of the confocal family) normal to either system of its circular sections, and so that in every position the sum of the moments of inertia of the entities $a_1 . M_1$, $a_2 . M_2$, . . . , with respect to it, is of any constant magnitude $2.s$; then will the envelope of the straight line $l\,l$ be a determinable quadric w of revolution, having the mean centre O as centre, and the fixed line $D D$ as axis. And all such quadrics w_1, w_2, w_3, . . . , corresponding to all possible values $2.s_1$, $2.s_2$, $2.s_3$, . . . , of the constant are determinable quadrics of revolution, having the mean centre O as common centre, and the line $D D$ as principal axis.

THEOREM 5.

The Locus of a straight line $l\,l$ through any fixed point D^1 in a line $D D$ through the mean centre O and normal to circular sections of the confocal quadrics Q_1, Q_2, Q_3, . . . , and such that the sum of the moments of inertia of the entities $a_1 . M_1$, $a_2 . M_2$, . . . , with respect to it, is of constant magnitude $2.s$, is a quadric cone of revolution. having the point D^1 as vertex, and D D as axis.

We know that the *locus* of the lines l l of intersection of all pairs of mutually orthogonal tangent planes to any quadric, cone C is another quadric, cone E concyclic with the reciprocal of the cone C. (See Salmon's "Geometry of Three Dimensions," Art. 247). And if C be a cone, such that the sum of the moments of inertia of the entities $a_1 . M_1$, $a_2 . M_2$, . . . , with respect to its tangent planes, severally, be equal to a constant s, we know that the sum of the moments of inertia of the entities with respect to the lines l l, severally, must be equal to $2.s$. Hence we have :—

Theorem 6.

Given any masses M_1, M_2, M_3, . . . , in space and corresponding numbers a_1, a_2, a_3, . . . , of known signs, as multipliers; the Locus of a straight line l l passing through any given point V in space, and such that the sum of the moments of inertia of the entities $a_1 . M_1$, $a_2 . M_2$, $a_3 . M_3$, . . . , with respect to it $=$ any constant $2.s$, is a quadric cone E, having the point V as vertex, and concyclic with the reciprocal of the cone C, having V as vertex, and such that the sum of the moments of inertia of the entities with respect to its tangent planes $= s$, &c.

Theorem 7.

If three planes, always mutually orthogonal, move in space so as to continue to be tangent planes respectively to any three of the confocal quadrics Q_1, Q_2, Q_3; then will the Locus of their common point of intersection be a *Sphere*, whose centre is coincident with the mean centre O of the entities $a_1 . M_1$, $a_2 . M_2$, . . . , which is also the centre of the quadrics.

Note.—This Theorem, which is an obvious deduction from the kinetic properties exposed, was arrived at by Salmon by means of a formula due to Chasles. (See Salmon's "Geometry of Three Dimensions," Art. 172.)

THEOREM 8.

If two planes A and B mutually orthogonal, be tangent planes respectively to any two quadrics Q_1, Q_2, of the confocal family; then will the other pair of tangent planes A^1 and B^1 through their line of intersection $l\,l$, to the same two quadrics, be mutually orthogonal.

This is an obvious deduction from the kinetic properties exposed.—The planes A and B being tangents to the quadrics Q_1 and Q_2, the moments of inertia of the entities $a_1 . M_1$, $a_2 . M_2$, . . . , with respect to them are constants s_1 and s_2; and the sum $s_1 + s_2$ of these moments of inertia is equal to the moment of inertia of the entities with respect to their line of intersection $l\,l$. And since the moment of inertia with respect to the line $l\,l$ is equal to the sum of the moments of inertia with respect to the tangent planes A^1 and B^2, it follows that A^1 and B^1 must be mutually orthogonal.

This theorem is an extension to confocal quadrics of one pertaining to confocal conics, due to Admiral De Jonquières of the French Navy, who is one of the most distinguished geometers in Europe. (See "Mélanges de Géométrie Pure," par E. De Jonquières.)

OBSERVATIONS.

The family of confocal quadrics Q_1, Q_2, Q_3, . . . , and the properties of inertia pertaining to them, are worthy of attention, not only on account of their intimate connection with "Wave Surfaces," and "Surfaces of Elasticity," but also on account of their direct applications to many important problems. (See Salmon's "Geometry of Three Dimensions," Arts. 467, 480, &c.)

2°.—Some interesting properties pertaining to confocal quadrics can be deduced by application of the numerous new theorems arrived at by the author, and published in Vol. X of the "Quarterly Journal of Pure and Applied Mathematics," under the title—"Properties of Quadrics having Common Intersection, and of Quadrics inscribed in the same Developable."

3°.—Since writing the present paper, the author has found that the question had been previously considered by the late Professor Townsend, of the Dublin University.

The results at which he arrived are given *without any investigations* on page 312 of Williamson's "Integral Calculus." From question 19, as there enunciated, it would appear that Townsend did not perceive that the envelope of the plane is an ellipsoid only when the prescribed moment of inertia is not less than a certain determinable magnitude; or that it is a Hyperboloid of One Sheet for all values less than such limiting value. Nor does it appear that he considered the case in which the envelope of the plane is a Hyperboloid of Two Sheets, or any limiting values of the moment of inertia.